Statistical Science

Chapter 2.6 Dimensions (from Schneider 2009 *Quantitative Ecology* Chapter 6).

ReCap (Ch 1)
Quantities (Ch2)
2.1 Five part definition
2.2 Types of measurement scale
2.3 Data collection, recording, and error checking
2.4 Graphical and tabular display of data
    Critique of graphs and tables
2.5 Ratio scale units
    Base units, derived units and standard multiples
    Unconventional units
2.6 Dimensions
    Euclidean
    Mechanical
    Composite
    Additional (Matroishkas, cash, *etc*)
    Entities (Chemical and Biological)
    Fractal

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Not here last time?
     Course Outline
     Name on roster
Questionnaire results
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Bring matroishkas
(similarity)
Bring maple leaves
(fractal objects)
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on chalk board

Recap Chapter 2
Quantities: Five part definition
Measurements made on four types of scale: nominal, ordinal, interval, ratio
Data collection, recording, and error checking
Graphical and tabular display of fully defined quantities
Units are useful in reasoning about quantities.
    Distinguish derived from base units, then define standard multiples.
    Unconventional units are useful science.

Today: Introduce concept of dimensions as a grouping of units,
    then develop concepts of composite dimensions and fractal dimensions,
Going to use units and dimensions as a means of reasoning about quantities.

Wrap-up:
Units are grouped by similarity into dimensions.
Fractal and composite dimensions are constructed relative to base dimensions.
Fractal dimensions and units are based on self-similarity.
Suggest that students spend next few days looking for examples of
Fractal vs Euclidean shapes.
    Fractal dimensions lie between line and plane, plane and volume.
    Euclidean shapes are anthropogenic. Fractal shapes are natural.

Why Dimensions?
Dimensions traditionally appear in introductory texts in physics and related disciplines, such as geophysics, hydrology, meteorology, and oceanography. They occur from time to time in physiology and biochemistry. They rarely occur in textbooks in biology and the environmental sciences. They are notably absent in statistical texts. The relevance and utility of dimensional analysis in statistical science are several.

Dimensions are the basis for unit conversion.
Dimensions are used to check the validity of equations expressing science concepts.
Dimensions are useful in understanding equations in science (Lab 2).
Dimensionless ratios apply across units and so facilitate insights about the relation of one quantity to another.
Statistical inferences are based on dimensionless ratios—the likelihood ratio and commonly used test statistics (*t, F,* X^2).

Definition: Dimensions are groups of similar units
Ratio scale units that are similar are grouped together into a <u>dimension</u> according to a principle of similitude. Thus, quantities measured in cm have the same dimensions as quantities measured in arm lengths, cubits, km, and nautical miles.

Euclidean Dimensions
These are related to one another by an integral change in exponent. The group (centimeters, meters, yards) is related to the group (centimeters^2, hectares, acres) by an increase in the exponent from 1 to 2. The first group has dimensions L^{+1}, the second group has dimensions L^2.

$$\left(\frac{10cm}{1cm}\right)^2 = \frac{10^2 cm^2}{1cm^2}$$

The length group (centimeters, meters, yards) is related to the volume group (centimeters^3, meters^3, yards^3) by an increase in the exponent from 1 to 3.

$$\left(\frac{10cm}{1cm}\right)^3 = \frac{10^3 cc}{1cc}$$

Consequently, simple fractions define the relation of any of the three Euclidean dimensions, such as area and volume.

$$\left(\frac{10cm^2}{1cm^2}\right)^{3/2} = \frac{10^{3/2} cm^3}{1cm^3}$$

Mechanical Dimensions
The mechanical dimensions are mass M, length L, and time T, one for each of the first three base units in the SI system.

Mass M	Length L	Time T
Kg	cm km	seconds
lbs	inch yd fathom	hours
µg	furlong	days
	rod chain	moons
	spearlength	lifetimes
	light-year	

Time, such as seconds, days, or millenia all belong to a single dimension symbolized by T for time. This use of the word "dimension" differs from that in which directions in an x y z coordinate system are all called different dimensions.

Composite dimensions
Many quantities have units with composite dimensions.

Examples: What dimensions do the following quantities have ?

area		
A = hectare = $(100\ \mathrm{m})^2$	$L \cdot L$	$= L^2$
$A = 15\ \mathrm{cm}^2$	$L \cdot L$	$= L^2$
volume		
$V = 1\ \mathrm{cm}^3 = 1\ \mathrm{cc}$	$L \cdot L \cdot L$	$= L^3$
velocity		
$\dot{x} = 15\ \mathrm{cm/sec}$	L/T	$= L^1\ T^{-1}$
respiration		
$\dot{V} = 15\ \mathrm{cc}\ O_2\ /\ \mathrm{sec}$	L^3/T	$= L^3\ T^{-1}$
kinetic energy		
$E = 15\ \mathrm{kg}\ (2\mathrm{cm/sec})^2 = 60\ \mathrm{km}\ \mathrm{cm}^2/\mathrm{sec}^{-2}$	$M \cdot (L/T)^2$	$= M\ L^2\ T^{-2}$

Example: convert several commonly used units (handout) to dimensions

The dimensions of mass, length, and time are not the only bases for composite dimensions. We could choose time T, area A, and energy E as our base dimensions. Within this system units of units of mass become a composite dimension of T^2E/A, while volume becomes a composite dimension of $A \cdot A^{1/2}$. Any grouping is valid as long as the units grouped into one dimension do not also belong to another dimension.

Temperature θ °Kelvin

For thermodynamics, including bioenergetics, the additional dimension is temperature, for which the symbol is θ
The standard unit is a kelvin K, one degree on the kelvin scale.

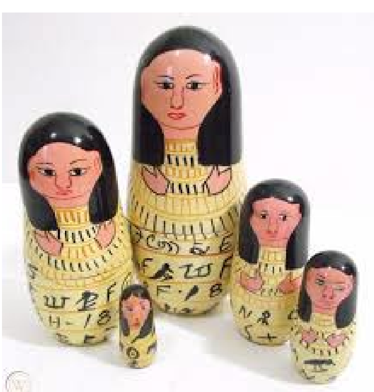
Temperature is a measure of heat content, and strictly speaking, we already have a composite unit for this $E/M = M^1\, L^2\, T^{-2}\, M^{-1} = L^2\, T^{-2} = \text{velocity}^2$
We can interpret temperature as the square of the velocity of the particles (atoms, molecules) and dispense with temperature as a separate dimension. However, this is awkward and inconvenient, so we treat temperature as a separate dimension, rather than as a squared velocity (derived dimension).

Charge Q coulomb

For electromagnetic quantities the standard unit is an ampere, which suggests that current would be the dimension. However, an ampere is a derived unit, 1 coulomb per second, where a coulomb is a mole of electrons.
Hence ampere = coulomb/second = $Q^1\, T^{-1}$
For electromagnetic quantities there is little need to introduce an additional dimension, if Q = charge with a standard unit of 1 coulomb (= 1 mol electrons).

Cash $ dollar dime megabuck peso

Additional dimensions
Depending on what we wish to measure, additional dimensions are defined by a notion of similarity. If we were interested in economics, we could define a dimension called "cash" with units of dollars, pennies, nickels, dimes, megabucks, *etc*. If we are interested in international economics, we could add foreign currencies: pesos, yen, francs, *etc*.



Additional dimensions

Matroishkas $\mathcal{M}$

Because the central idea of a dimension is that quantities are grouped according to some notion of similarity, we can use any form of similarity that we like. We could, if we like, define a dimension called "matroishkas" instead of length. This would consist of all measuring units shaped like matroishkas (nesting dolls). These objects all have the same shape.

Entities

Chemical Entities #
mol
millimol
micromol
nanomol
picomol

For chemistry and biochemistry, we add the dimension of recognizable chemical entities (Stahl 1962). Examples of chemical entities are atoms, ions, and molecules.

The dimension of chemical entities does not have a conventional symbol. One choice is #, for which the standard unit is the mole.
1 mol = $6.02 \cdot 10^{23}$ particles

In biology, the additional dimension will be a count of recognizable biological entities (Stahl 1962). It is useful to distinguish biological entities at different <u>levels</u> of biological organization.

Biochemical entitites: ions, atoms, molecules (including proteins)

Genetic entities: chromosomes, genes, alleles, mutations

Cellular entities: nuclei, mitochondria, cells

Behavioural entities: attempts, successes, modal action patterns (MAPs)

Population entities: interacting species

Community level entities: number of taxa (species, order, *etc*), number of trophic levels.

Biological Entities #
mol
dozen
gross
score
kilocount
megacount

There is no standard symbol or standard units for the dimension of biological entities. We could again use Q for this dimension, with a standard unit of a mole. However, this is going to be inconvenient, as we are often interested in exact counts of small numbers of entities.

Possible units:

2 mol bacteria	$= 2 \cdot 6.02 \cdot 10^{23}$ bacteria	inconveniently large
2 dozen genes	$= 2 \cdot 12$ genes	inconvenient, not divisible by 10
2 gross alleles	$= 2 \cdot 12$ dozen alleles	also inconvenient
2 score cells	$= 2 \cdot 20$ cells	
2 kilocount ants	$= 2 \cdot 10^3$ ants	
2 megacount fish	$= 2 \cdot 10^6$ fish	

In biology, the most useful units are kilocounts, megacounts, or variants:
For example, a protein might measure 120 kilobases long.

Units such as a kilocount of cells, or a kilocount of predator attacks, or a megacount of potential encounters are not standard, but they are useful in biology and can be handled in a rigorous fashion (Stahl 1962). The philosophical objection to using counts of objects or events as a measurement scale (Ellis 1966) can be met by insisting that this scale does not consist of numbers; it has units of entities (animals, genes, *etc*) counted on a ratio scale. This reasoning follows Kyburg (1984), who argues that all measurements must have units.

Dimensionless Quantities

The ratio of two quantities with the same units is a dimensionless number

Name	Symbol	Explanation
Ratio of like quantities	Q/Q_{ref}	Q and Q_{ref} have the same units
Relative variation	$\Delta Q/Q$	ΔQ is difference in two values
Relative difference	$d \ln Q = Q^{-1}\, d\, Q$	
Doubling ratio	$\log_2 (Q/Q_{ref})$	
e-fold logarithmic ratio	$\ln (Q/Q_{ref})$	
Ten fold logarithmic ratio	$\log_{10} (Q/Q_{ref})$	
Binomial ratio	n+ / N	n+ is number of successes, N is number of trials
Probability of an event	$\Pr(X{=}x \mid \theta)$	X is variable, real number x on the interval $0 \leq x \leq 1$
		θ is known parameter
Likelihood ratio *LR*	$L(\theta \mid X) / L(\theta_{ref} \mid X)$	θ is estimate of unknown parameter, given data X
Support	ln *LR*	Evidential support for θ relative to θ_{ref}

In this course we will be calculating ratios--Odds ratios, Likelihood ratios, and statistical ratios (t, F, X^2) that are used to calculate probabilities.

Dimensions (concluded)

Reasoning according to the principle of dimensional similarity has a long history.
It goes back to Galileo and Newton (in physics), to Fourier (in thermodynamics), and to D'Arcy Thompson (in biology).
Dimensional or similarity arguments have an important place in science.

Dimensions are a way of thinking about quantities based on similarity.

Which ones are similar? Which are related ?

Example: what are the dimensions of a flux ?

Diagram of particles moving laterally through a plane

Diagram of particles in a cubical volume with arrow showing lateral motion

Flux = seed number drifting laterally

Dimensions are: $\# L^{-2} T^{-1}$ = density /time

Flux = seed concentration · velocity

$= \#/V \cdot L/T = \# L^{-2} T^{-1}$

These two fluxes appear to be different, because drawn in different ways, and perhaps measured differently.

But they are equivalent. They have the same dimensions.

Mass Flux = mass of seeds drifting laterally = $M\, A^{-1}\, T^{-1}$

This is not the same as numerical flux.

Example: flux of nutrients across cell wall. What units ? (typical)

This illustrates quantitative reasoning based on grouping units according to <u>similarity</u>.

Fractal dimensions

Euclidean dimensions are related to one another by integral powers (exponents).

$$\left(\frac{10cm}{1cm}\right)^3 = \frac{10^3 cc}{1cc}$$

Fractal dimensions are related to one another by fractional exponents.
For example, we can have a dimension of crooked lengths L^D where
$1 \le D \le 2$ "D between 1 and 2"

$$\left(\frac{100m}{1m}\right)^{1.3} = \frac{100^{1.3} m^{1.3}}{1m^{1.3}}$$

What does a dimension "between 1 and 2" mean ?
The units in this dimension are all more convoluted than a straight line ($D_f = 1$), but not so convoluted as to fill a plane ($D_f = 2$).
All of the units in this dimension are equally convoluted.

If $D_f = 1.4$, then we have $m^{1.4}$ $km^{1.4}$ $yd^{1.4}$ $ft^{1.4}$ etc

Example of maple leaves. Each person gets a leaf.

Look at perimeter from far away (hold up leaf).
You can see perimeter is convoluted into series of 5 major lobes.
Now look more closely.
You can see that within each major lobe there is more convolution.
It turns out that the degree of convolution within each major lobe is nearly the same as the degree of overall convolution.
If you look even more closely, you can see that extremely fine serrations exist on the minor lobes within the major lobes.
The perimeter of the leaf is of similar convolution at large, medium, small scales, and all scales in between.
This idea is quantified as a fractal dimensions L^{Df}

We can extend this to fractal areas for L^{Df} where $2 \le D_f \le 3$

Another example: The sea surface on a calm day. It is nearly flat, $D_f = 2$.
Then wind picks up, creating small waves, which begin developing into larger waves, so that after a time we have small waves on medium waves on large waves. The dimension of the sea surface has increased from L^2 to something more convoluted, say $L^{2.2}$. We could measure this in fractal $m^{2.2}$ fractal $km^{2.2}$ *etc*.

Fractal Dimensions in Science

We are thoroughly familiar and take for granted the logic of the Euclidean world of street grids, buildings, walls, floors, tables, and plates. But we ourselves are fractal. Our lungs, blood vessels, and nervous systems are fractal. We live on a planet with fractal landscapes, formed by fractal rivers. We have been taught to think according to Euclidean dimensions, while living in a fractal world.

When we alter the landscape we tend to reduce its fractal dimension. We building straight roads with simple curves. We straighten rivers and lay out fields with straight boundaries. The undisturbed landscape remains fractal.

The habitats that support life are fractal. A convoluted shoreline, being fractal, provides more habitat to bacteria and fish than does a seawall. In addition, a fractal shoreline provides more habitat to bacteria than to fish. Bacterial can be laid end to end along a stretch of seafloor than can fish. Bacteria end to end fall along a longer length than fish end to end along the same stretch. If we straighten out the chain of bacteria and of fish, the chain of bacteria will exceed the length of the fish chain. The concept of fractal dimension permits us to compute how much longer the bacteria chain will be, based on the fractal dimension and the ratio of the length of the fish to a bacterium.

Fractals are a non-intuitive yet completely appropriate way of measuring natural objects. They allow us to quantifying the complexity of natural objects--cells, tissues, organisms, populations, habitats, and ecosystems.

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Suggest that students try looking at their
surroundings for fractal rather than
Euclidean shapes.
I.e., surfaces more convoluted than a flat
  plane(dimensions greater than 2 butless
  than 3).
I.e. lines move crooked than a straight line
  (dimension somewhere between Length m¹ and
  area m²).
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Euclidean and Fractal Dimensions -- References

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<The original book on the topic, still worth reading.>

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<short, highly readable account, including how to estimate km^d>

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